Unit C

Suggested Answers and Guidelines

Suggested Answers and Guidelines for Unit C

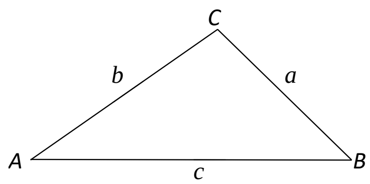
Pre-lesson Worksheet 3

Cosine Formula

Sine Formula

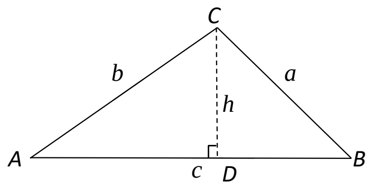
Given a triangle ABC, let $a = BC$, $b = AC$, $c = AB$. There is a nice relationship between the sides and angles:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Proof Let D be a point on AB such that $CD \perp AB$.

Let $h = AD$. Then, in ΔACD, we have



$$\sin A = \frac{h}{b}$$
$$h = b\sin A$$

In ΔBCD, we have

$$\sin B = \frac{h}{a}$$
$$h = a\sin B$$

Hence comparing the two expressions of h,

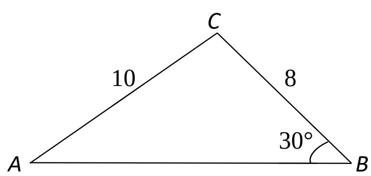
$$a\sin B = b\sin A$$
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Similarly, we have $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Exercise

Find the angles *A* and *C*, and the side *AB*, using the sine law.

[Ans: $A = 23.6°$, $C = 126°$, $AB = 16.1$]



To find A,

$$\frac{8}{\sin A} = \frac{10}{\sin 30°}$$
$$\sin A = \frac{8\sin 30°}{10}$$
$$A = 23.57817848°$$

Then, $C = 180° - 30° - A = 126.4218215°$

To find AB,

$$\frac{AB}{\sin 126.4218215°} = \frac{10}{\sin 30°}$$
$$AB = 16.09335462$$

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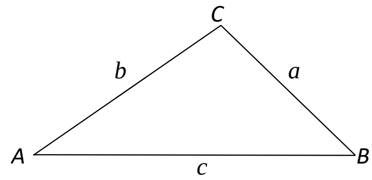
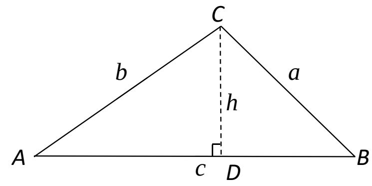
<u>Area formula for a triangle</u>

From the proof of the sine law, we can deduce a useful formula for computing the area of a triangle, namely:



Area of $\Delta ABC = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$

Proof With the setup from the proof of sine law, we can deduce from ΔACD that



$h = b\sin A$ (express in terms of b and angle A)

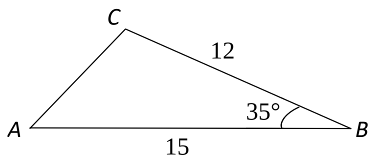
Hence the area of ΔABC

$$= \frac{1}{2}\times AB\times CD = \frac{1}{2}ch = \frac{1}{2}c(b\sin A) = \frac{1}{2}bc\sin A$$

The other two forming the area can be proved similarly. *(end of proof)*

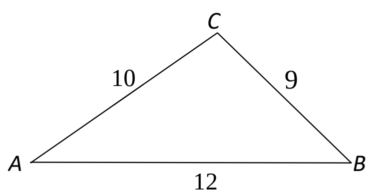
<u>Exercise</u>

Find the area of ΔABC in the figure. [Ans: 51.6]



$$Area = \frac{1}{2}(12)(15)\sin 35° = 51.6$$

Think: If the three sides of a triangle are given, but none of the angles are given, can you still find the angles?



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Lesson Worksheet 3

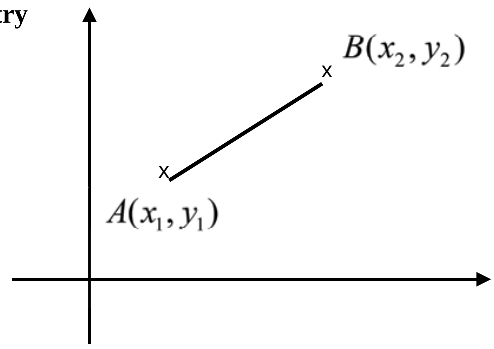
Task 1: Proof of Cosine Formula using Coordinate Geometry

Distance Formula

Given two points (x_1, y_1) and (x_2, y_2) in a coordinate.

The distance between the points is given by the formula

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



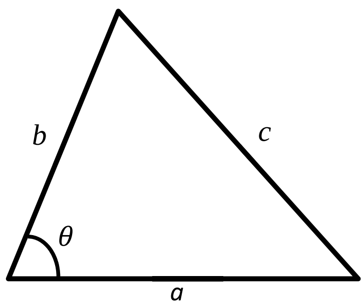


Fig. 1a

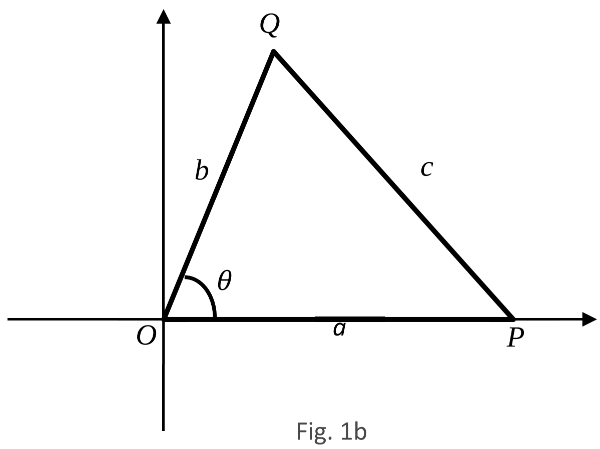


Fig. 1b

Fig. 1a shows a triangle with sides a , b and c, with the angle θ is opposite to side c .

This triangle is placed on a rectangular coordinate plane as shown in Fig. 1b.

O is the origin and the coordinates of P are $(a, 0)$.

1. Express the coordinate of Q in terms of b and θ .

$$Q = (b\cos\theta, b\sin\theta)$$

2. By the distance formula,

$$c = \sqrt{(a - b\cos\theta)^2 + (0 - b\sin\theta)^2}$$

Square and expand both sides of the equation :

$$c^2 = (a - b\cos\theta)^2 + (-b\sin\theta)^2$$
$$c^2 = a^2 - 2ab\cos\theta + b^2\cos^2\theta + b^2\sin^2\theta$$
$$c^2 = a^2 + b^2(\cos^2\theta + \sin^2\theta) - 2ab\cos\theta$$
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

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Yes, under the definition of trigonometric function of general angle. The proof works even when θ is an obtuse angle as the coordinates of Q is still $(b\cos\theta, b\sin\theta)$.

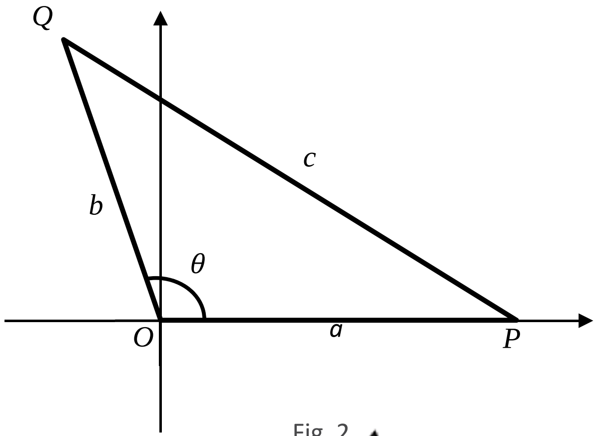

Fig. 2

Task 2: Using trigonometry

Fill in the details of the proof based on Fig. 3
Drop the perpendicular onto the side c ,

$$c = \_\_\_\_\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$c = a\cos\beta + b\cos\alpha$$

Multiply through by c to get

$$c^2 = ac\cos\beta + bc\cos\alpha$$

By considering the other perpendiculars, it obtains

$$a^2 = ac\cos\beta + ab\cos\gamma \ ,$$

$$b^2 = bc\cos\alpha + ab\cos\gamma$$

Adding the latter two equations gives

$$a^2 + b^2 = ac\cos\beta + bc\cos\alpha + 2ab\cos\gamma$$

Subtracting the first equation from the last one we have

$$a^2 + b^2 - c^2 = -ac\cos\beta - bc\cos\alpha + ac\cos\beta + bc\cos\alpha + 2ab\cos\gamma$$

which simplifies to

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

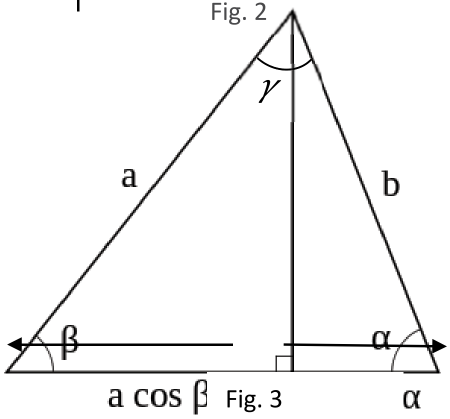

Fig. 3

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Task 3: Using the Pythagorean theorem

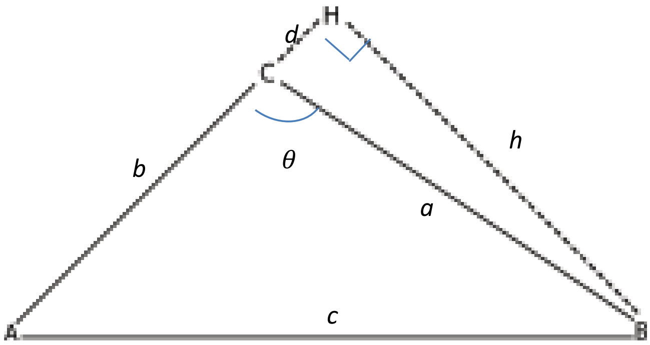

Fig. 4a – Obtuse triangle ABC with height BH

Case of an obtuse angle

Using d to denote the line segment CH and h for the height BH, triangle AHB gives us

$$c^2 = (b+d)^2 + h^2$$

and triangle CHB gives

$$d^2 + h^2 = a^2$$

Expanding the first equation gives

$$c^2 = b^2 + 2bd + d^2 + h^2$$

Substituting the second equation into this, the following can be obtained:

$$c^2 = a^2 + b^2 + 2bd$$

Note that $d = a\cos(180° - \gamma) = -a\cos\gamma$

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Case of an acute angle

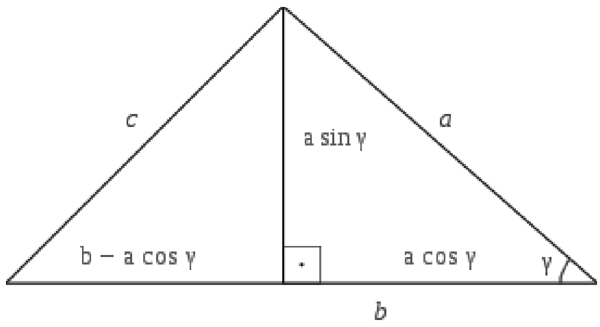

Fig. 4b – A short proof using trigonometry for the case of an acute angle

Another proof in the acute case

Using more trigonometry, the Cosine Formula can be deduced by using the Pythagorean theorem only *once*. In fact, by using the right triangle on the left hand side of Fig. 4 it can be shown that:

$$c^2 = (b - a\cos\gamma)^2 + (a\sin\gamma)^2$$

$$= b^2 - 2ab\cos\gamma + a^2\cos^2\gamma + a^2\sin^2\gamma$$

$$= b^2 + a^2 - 2ab\cos\gamma,$$

If $b < a\cos\gamma$, how will you modify the above proof?

In this case, the right triangle to which the Pythagorean theorem is applied moves *outside* the triangle ABC . The only effect on the calculation is that the quantity $b - a\cos\gamma$ is replaced by $a\cos\gamma - b$. As this quantity enters the calculation only through its square, the rest of the proof is unaffected. However, this problem only occurs when β is obtuse, and may be avoided by reflecting the triangle about the bisector of γ .

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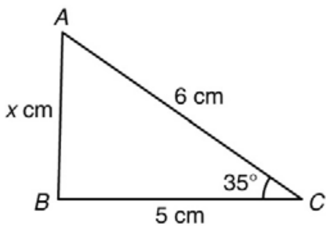
Lesson Worksheet 4

Cosine Formula

Practice on Cosine Formula (Give your answers correct to 3 significant figures.)

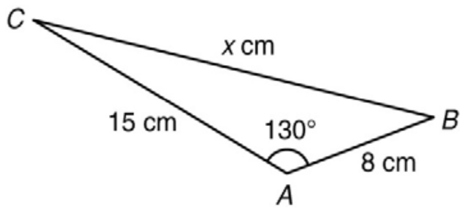
1. Find x.



By the cosine formula,

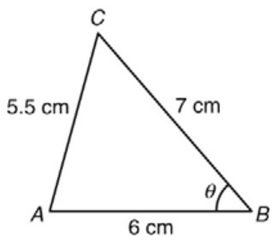
$$x^2 = 5^2 + 6^2 - 2(5)(6)\cos 35°$$
$$x = 3.44$$

2. Find x.



$$x^2 = 15^2 + 8^2 - 2(15)(8)\cos 130°$$
$$x = 21.1$$

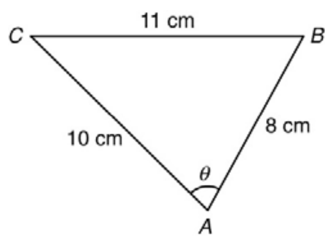
3. Find θ.



By the cosine formula,

$$\cos\theta = \frac{6^2 + 7^2 - 5.5^2}{2(6)(7)}$$
$$\theta = 49.3°$$

4. Find θ.



$$\cos\theta = \frac{8^2 + 10^2 - 11^2}{2(8)(10)}$$
$$\theta = 74.4°$$

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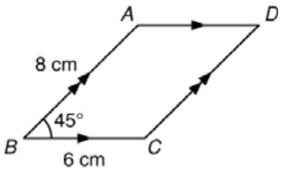
<u>Harder Questions</u>

5. In the figure, $ABCD$ is a parallelogram.

Find the length of the diagonal BD.



$\angle A = 180° - 45° = 135°$

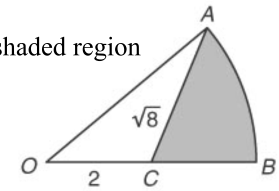
$BD^2 = 6^2 + 8^2 - 2(6)(8)\cos 135°$

$BD = 13.0$

6. OAB is a sector of a circle with centre O. C is the mid-point of OB such that $CA = \sqrt{8}$ and $OC = 2$. Find

(a) $\angle AOB$,

(b) the area of the shaded region



$BC = 2$, $OA = 4$

$$\cos \angle AOC = \frac{2^2 + 4^2 - (\sqrt{8})^2}{2(2)(4)}$$

$\angle AOC = 41.40962211°$

$$Area = \pi(4^2)\frac{41.40962211°}{360°} - \frac{1}{2}(2)(4)\sin 41.40962211°$$

$$= 3.14$$