賽馬會「知優致優」計劃
Jockey Club "Giftedness Into Flourishing Talents" Project

Isoperimetric Problem

Mathematics Secondary 2

Level 2: School-based Pull-out Programme

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Background and Notes

The design of the learning and teaching plan reflects the actual circumstances of the particular school at the time of implementation. As it is developed and tailor-made to meet the specific cognitive and affective needs of students, all learning and teaching resources are for reference only.

When adapting the materials, curriculum, instructional and assessment modifications can be made in accordance with the diverse needs and abilities, learning styles and aspirations of students, professional competence of teachers, and gifted education development of the schools.

Teachers are strongly recommended to read the introduction, theoretical background and summary of the resource package to have a better understanding of the principles of Gifted Education and strategies for implementation.

This unit includes 1 foreword, 1 lesson plan, 10 sets of worksheets, 5 sets of suggested answers and 1 set of extension materials.

With reference to our resources, educators can design suitable learning activities and implement the elements of Gifted Education, based on students' needs and interests, and teaching experience, so as to unfold students' potentials to the fullest.

Isoperimetric Problem

Grade: Secondary 2
No. of Lessons (Learning Time): 7 Units (60 -75 minutes for each unit) + 1 Extension Unit

Operation Mode of Gifted Education	Level 2: School-based Pull-out Programme
Target Students	- S2 students with outstanding performance, rigorous reasoning skills and high comprehension power in Mathematics - Students having a strong interest in Mathematics, for the extension of learning after school and pull-out class - Students having a strong task commitment in Mathematics, for the high perseverance in solving challenging problems

Foreword / Background

When teaching the topic “Area and Perimeter” in the S1 curriculum, teachers had introduced the following results using computer simulation: when the perimeter of a n-sided polygon is fixed, the polygon attains its maximum area when it is a regular polygon. However, the proof or the reason behind was not discussed in the lesson as it involved advanced content knowledge and proving skills not yet learnt by most students. Such knowledge and skills are excellent elements for a pull-out programme, which aims to provide enrichment knowledge and training of thinking skills for students with outstanding performance and strong interest in Mathematics.

The Project School has a tradition in which the S2 Mathematics pull-out programme is led by a few student instructors studying in S5 in the school. At the beginning of this programme, the teacher-in-charge discussed and designed the course content with the Project GIFT team and developed the learning materials. Student instructors then studied the materials to gain understanding and to think of guiding questions for the S2 participants. Student instructors were also responsible for teaching the pull-out class with some support from the teacher-in-charge. The teacher had a discussion with student instructors after each lesson to help them reflect and improve their teaching performance and confidence.

Objectives of Collaboration

The aim of the collaboration was to develop a pull-out gifted Mathematics programme which can serve as an extension of the regular curriculum. The programme provides a detailed discussion of the solution to the Isoperimetric Problem, a well-known Mathematical problem with a long history. The solution is broken down into a lot of short propositions to be proved. Through proving these propositions, a wide range of Mathematical thinking and proving skills, such as generalization and proof by contradiction, can be introduced to students. In order to prove the propositions, students will also need to attain Mathematical knowledge above their level.

S2 students are expected to gain a deeper understanding in Mathematics and develop their higher-order thinking skills, especially the skills of constructing different types of Mathematical proofs and giving judgements with rigorous reasoning. Through discussion, presenting their solutions and challenging others' solutions, students can develop social and presentation skills. Students can also get used to comprehending materials of advanced levels and gain better self-learning ability.

S5 student instructors can also benefit from the programme. They need to comprehend unfamiliar Mathematics content knowledge. When preparing and conducting the lesson, they may need to convert the given reading materials into an interactive lesson which is rich in questions and discussions. They gain valuable experience of teaching Mathematics, which strengthens their personal-social competence and leadership skills. Moreover, to manage the extra workload as an instructor, they are expected to develop better time-management skills. All these skills are important for their future careers.

Student Selection Criteria and Procedures

Teachers were advised to adopt multiple approaches to select students. Beside looking at results in Mathematics examinations and competitions, a short screening test was also given to all S2 students. Questions in the screening test involved non-routine problems selected from Mathematics competitions to better assess students' problem solving ability. Previously, students' logical reasoning ability had been assessed with the help of Project GIFT. These test results could be reviewed during the selection process. Furthermore, Mathematics teachers of each S2 class could nominate students based on their observation in the lesson. Finally, there were also opportunities for students to nominate themselves and join the programme.

Theoretical Framework

Gavin and Sheffield outlined four components that characterize an advanced and in-depth curriculum appropriate for gifted Mathematics students (Gavin, 2016):

1. Creative and complex problem-solving
2. Connections within and access across Mathematical and other content areas and across a wide range of contexts
3. An inquiry-based approach that focuses on processes used by Mathematicians
4. Appropriate pacing

An advanced and in-depth curriculum should involve both above-grade-level content knowledge and deep Mathematical investigations that are challenging for gifted students. Creative and complex problems refer to those problems that require students to think at high levels, to test out ideas, to discuss ideas and to demonstrate their reasoning skills. Through an inquiry-based approach, students are encouraged to explore and discover Mathematics knowledge. In a gifted programme, an appropriate pacing is to keep repetition and practice at a minimum when introducing new materials and provide enough time to discuss, struggle and reflect during investigation and problem-solving.

Proof and proving can deepen Mathematical understanding and broaden reasoning skills. Mathematical proof consists of not only a sequence of steps following formal notation and rules, but also a sequence of ideas and insights (Hanna & Villiers, 2008). Through studying and writing Mathematical proofs, students practice building a sequence of logical steps and applying Mathematical ideas they have known to construct arguments. These processes help students to achieve Mathematical understanding (Hanna & Mason, 2014). Thus, proof and proving is an important theme in Mathematics and the nature of Mathematical proof also coincides with the advanced and in-depth curriculum for gifted Mathematics students.

Learning and Teaching Strategies

A peer tutoring approach has been the established practice of the school over the years. In each lesson, student instructors take turns to lead the discussion with the participating students. Besides, they also walk through each group of participants and provide hints to them. With suitable guidance and supervision from teachers, the approach can provide a platform for exchanging and building up Mathematical knowledge among junior and senior form students. Leadership skills of the instructors are also utilized when taking a teacher's role in the programme.

In the first two units, content knowledge about trigonometry, area formula of triangles and the method of completing the squares are introduced to students through direct demonstration by student instructors or the teacher. Most of this content can be found in S3 and S4 Mathematics textbooks. A suitable approach to teach such knowledge is to keep drilling practice at a minimum and to spend much time delivering the reasoning behind the proofs of the formula.

In the next five units, students are given pre-lesson materials to encourage self-learning and to equip them with the knowledge necessary for the lesson. In the lesson, they are given a few problem-solving tasks which require students to prove or disprove a statement. Students work in groups to discuss and solve the problems. Student instructors provide hints and guiding questions (See Lesson Worksheets) from time to time to help students. The approach follows the theoretical framework that the pull-out programme should involve complex problem-solving and an inquiry-based approach. Finally, student instructors help summarize the proving skills or inquiry skills related to the problems.

Learning Content and Activities

The course content covers a few results about maximizing the area of figures. It involves knowledge of senior form Mathematics. These learning contents can finally lead to the result that 'If the perimeter of a polygon is fixed, the area of the polygon is maximum when it is regular.' With the concept of limit, this result can finally explain why the circle encloses the greatest area among all the closed curves in a plane. The programme provides an approach to explain the isoperimetric problem. Through this programme, a few Mathematical thinking and proving skills are highlighted. The programme is divided into 7 units corresponding to 7 lessons of about 60 to 75 minutes, plus an extension unit which is not covered in lesson time.

Unit	Preliminary Knowledge
A & B	- Trigonometric Ratios of Triangles (S2 Level) - Trigonometric Identities (S2/S3 Level) - General trigonometric functions (S4 Level) - Area Formula : 1/2 ab sin θ (S4 Level) - Heron's Formula (S4 Level) - Completing Square (S4 Level) In the lessons, teacher goes through the above methods or theorems with a few drilling exercise. Lesson Plans and materials are not provided. Teacher can refer to regular curriculum content.

Unit	Pre-lesson Content	Content during / after Lesson	Skills Highlighted
C	Graph of saine and cosine functions	**Triangle and Rectangle** Under fixed perimeter, (A) if the length of two sides of a triangle is fixed, the area is maximum when it is a right-angled triangle. (B) if the length of one side of a triangle is fixed, the area is maximum when the rest of the two sides have equal lengths. (C) square has the largest area among rectangles	Using Formula Algebraic Method (Completing the Square)
D	Triangle Inequality Quadrilateral Inequality	**From Triangle to Quadrilateral** (D1) Given 3 lengths, only one unique triangle can be formed. (D2) Given 4 lengths, more than one can be formed. (E1) For any triangle, there exists a circle the three vertices of the triangle. (E2) For some quadrilaterals, there does not exist a circle passing the four vertices of the quadrilateral.	Generalizing results Disproof by counter-example

Unit	Pre-lesson Content	Content during / after Lesson	Skills Highlighted
E	Angle at centre twice angle at circumference Opposite angles of cyclic quadrilateral	**Property of Circle** (F) A quadrilateral is cyclic if opposite angles are supplementary. (G) A problem related to angle at centre twice angle at circumference.	Separate proof into cases Proof by Contradiction
F	Angles in the same segment Problem (B) revisited	**Polygon (1)** (H) For any concave polygon, a convex polygon can be constructed so that the perimeter is unchanged but the area is greater. (Assumption) Under fixed perimeter and number of sides, there exists a polygon having the greatest area. (I) Under fixed perimeter, area of equilateral polygon is larger than that of polygon.	Suggesting a geometric construction Proof by Contradiction
G	Equilateral cannot imply regular Bretschneider's Formula	**Polygon (2)** (J) Given 4 lengths of a quadrilateral, the area is maximum when it is a cyclic quadrilateral. (K) Final Result: Prove the area of regular polygon is larger than that of equilateral polygon under fixed perimeter.	Using Formula Proof by using previous results
H (Extension)	**Isoperimetric Problem** Supplementary materials are provided for self-study. Area formula for a regular polygon, Limit, Isoperimetric Problem and its importance.		

Discussion

Three lessons (Units 4, 5 and 7) of the programme were observed by the Project GIFT team. It was observed that student participants were keen on learning Mathematics and had strong task commitment. For Unit 4, student participants could acquire the skills of disproving a statement by giving counter-examples. Some of them could give more than one counter-example. For Unit 5, students were puzzled about the idea of 'Proof by Contradiction' as this type of proof seldom appeared in the general Math curriculum. However, students still wrote down the notes and some of them stayed behind to ask the instructors after the lesson. For Unit 7, which is the last lesson, the instructors asked students to recall the knowledge they had learnt in the programme. Some students reflected that they had learnt content knowledge related to trigonometry, properties of circle and area formula. Some students moved on to answer they had learnt the way to disprove a statement by counter-example and a new type of proof called proof by contradiction. All these responses provided evidence that both knowledge and thinking skills were delivered to students effectively through the programme.

In each of the three lessons observed, the instructors took turns to lead the discussions while presenting the learning points and engaging students in discussions. They were well-prepared as they provided suitable hints and helped learners acquire the content knowledge with a variety of questions. They also responded to students' answers and questions properly and brought in further discussion. They insisted on the rigorous nature of Mathematics and helped students to justify their answer with strong reasoning. On top of that, they all adopted a serious attitude towards taking forward the lesson, they also demonstrated genuine interest in learning math and the joy of having a problem solved. As a result, student learners were mobilized and kept on-task. Their performance was highly appreciated by both the Project GIFT team and the teacher-in-charge of the pull-out programme.

Throughout the implementation of the programme, students and instructors could handle the learning tasks very well except the tasks related to proof by contradiction, which is the major focus of Unit 5 and Unit 6. It is advised to involve discussion about the negation of 'If-then statements' in Unit 5 and Unit 6. Also, teachers or instructors can introduce some more proofs by contradiction such as the proofs for 'There are infinitely many primes', '$\sqrt{2}$ is irrational' and 'Converse of Pythagorean Theorem'.

Overall, the Project School displayed a way to start a student-led pull-out programme in this programme. Teachers set the major focus of the whole programme and designed the learning materials of each lesson. High-ability senior form students with good presentation and leadership skills were chosen as instructors. To train them, they were asked to understand the course content through self-learning. In the first few lessons, teachers demonstrated skills in raising questions and leading whole-class discussions in a pull-out class. In the later lessons, student instructors took the role as instructors. Teachers might only help teach a small section or help the instructors by giving suggestions after the lesson. In the long run, the instructors could become mature and confident enough to hold the class without teacher assistance. The student instructors in the project school reflected that they were delighted to see their own improvements in driving the atmosphere of the lesson, interacting with students, identifying students' learning difficulties and managing time. These reflections provided evidence that both student participants and instructors could benefit through peer-mentoring.